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J. Math. Anal. Appl. 282 (2003) 56–62
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A generalization of discrete Gronwall inequality and its application to weakly singular Volterra integral equation of the second kind

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Received 13 December 2001
Submitted by R.P. Agarwal

Abstract

This paper presents a new discrete Gronwall inequality. Using the inequality, we prove convergence and error estimate of the numerical solutions of the second weakly singular Volterra integral equation, where discrete equation is derived by Novot's quadrature formula.

1. Introduction

If a nonnegative sequence $\{y_n,\ n = 0, \dots, N\}$ satisfies

$$y_0 = 0, \qquad y_n \leqslant A + Bh\sum_{j=0}^{n-1} y_j, \quad 1 \leqslant n \leqslant N,\ h = 1/N, \tag{1}$$

then

$$\max_{0\leqslant i\leqslant N} y_i \leqslant Ae^B, \tag{2}$$

where A and B are positive constants independent of h. As a discrete analogue of Gronwall inequality [5] this inequality is called discrete Gronwall inequality, which plays

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0022-247X/03/$ – see front matter
doi:10.1016/S0022-247X(02)00369-4

an important role in proving convergence of numerical solutions of Volterra integral equations with a continuous kernel [3,4,9]. But it is ineffective to prove convergence of numerical solutions of Volterra integral equations of the second kind with a weakly singular kernel. For example, we consider the following nonlinear Volterra equation of the second kind

$$u(s) = y(s) + \int_a^s k^*\big(s, t, u(t)\big)\,dt, \quad a \leqslant s \leqslant b, \tag{3}$$

where the kernel

$$k^*\big(s, t, u(t)\big) = (s-t)^\alpha \big(\ln|s-t|\big)^\beta k\big(s, t, u(t)\big), \quad -1 < \alpha \leqslant 0,\ \beta = 0, 1, \tag{4}$$

is weakly singular and $k(s, t, u(t))$ is a continuous function on variables s, t, u, especially, there exits a positive constant L satisfying

$$\big|k(s, t, u) - k(s, t, v)\big| \leqslant L|u - v|, \quad a \leqslant t, s \leqslant b. \tag{5}$$

Moreover for fixed s and t, we assume that $k(s, t, u(t))$ has high order derivatives on u, and let $k_u(s, t, u(t)) = \frac{\partial}{\partial u}k(s, t, u(t))$. In order to get a discrete equation of (3), we can apply Navot's quadrature formula of computing integrals with the end point singularity. Consider the integral

$$I(G) = \int_a^b G(x)\,dx = \int_a^b (b-x)^\alpha \big(\ln|b-x|\big)^\beta g(x)\,dx. \tag{6}$$

where $-1 < \alpha < 0$, $\beta = 0, 1$, and $G(x) = (b-x)^\alpha(\ln|b-x|)^\beta g(x)$, $g(x)$ is smooth on $[a, b]$. Take the step width $h = (b-a)/N$, and $x_i = a + ih$, $i = 0, \ldots, N$. If $g(x) \in C^{2m}[a, b]$, then Navot [7] and Lyness [6] proved that the quadrature formula

$$Q_N(G) = \frac{h}{2}G(x_0) + h\sum_{i=1}^{N-1} G(x_i) - \big[-\beta\zeta'(-\alpha) + \zeta(-\alpha)(\ln h)^\beta\big]g(b)h^{1+\alpha} \tag{7}$$

possesses the following Euler–Maclaurin asymptotic expansion

$$\begin{aligned} E_N(G) = Q_N(G) - I(G) = {} & \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!} G^{(2j-1)}(a)h^{2j} \\ & + \sum_{j=1}^{2m-1} (-1)^j\big[-\beta\zeta'(-\alpha - j) + \zeta(-\alpha - j)(\ln h)^\beta\big] \\ & \times g^{(j)}(b)h^{j+\alpha+1} + \mathrm{O}\big(h^{2m}\big), \end{aligned} \tag{8}$$

where B_{2j} are Bernoulli numbers, and $\zeta(x)$ is the Riemann Zeta function. From (8) we know that

$$E_N(G) = \mathrm{O}\big(h^{2+\alpha}|\ln h|^\beta\big), \tag{9}$$

if $g(x) \in C^2[a,b]$. Taking $s = x_i$ in (3) and using the quadrature formula (7) for

$$u(x_i) = y(x_i) + \int_{x_0}^{x_i} (x_i - t)^\alpha \big(\ln|x_i - t|\big)^\beta k\big(x, t, u(t)\big)\,dt, \tag{10}$$

we obtain the following nonlinear discrete equations: find u_i, $i = 0, 1, \ldots, N$, satisfying

$$\begin{cases} u_0 = y(x_0), \\ u_i = y(x_i) + \frac{h}{2}(x_i - x_0)^\alpha \big(\ln|x_i - x_0|\big)^\beta k(x_i, x_0, u_0) \\ \qquad + h\sum_{j=1}^{i-1}(x_i - x_j)^\alpha \big(\ln|x_i - x_j|\big)^\beta k(x_i, x_j, u_j) \\ \qquad - \big[-\beta\zeta'(-\alpha) + \zeta(-\alpha)(\ln h)^\beta\big] k(x_i, x_i, u_i) h^{1+\alpha}, \quad i = 1, \ldots, N. \end{cases} \tag{11}$$

But by means of (8) the integral equation (3) can be expressed as follows

$$\begin{aligned} u(x_i) = y(x_i) &+ h\sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j|\big)^\beta k\big(x_i, x_j, u(x_j)\big) \\ &+ h w_{ii} k\big(x_i, x_i, u(x_i)\big) + E_{i,t}\big((x_i - t)^\alpha \big(\ln|x_i - t|\big)^\beta k\big(x_i, t, u(t)\big)\big), \\ i = 0, 1, \ldots, N, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &w_{i0} = \frac{1}{2}, \qquad w_{ii} = h^\alpha\big[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta\big], \\ &\text{and} \quad w_{ij} = 1, \quad \text{for } 1 \leqslant j < i, \end{aligned} \tag{13}$$

and the remainder has an estimation

$$\Big|E_{i,t}\big((x_i - t)^\alpha \big(\ln|x_i - t|\big)^\beta k\big(x_i, t, u(t)\big)\big)\Big| = \mathrm{O}\big(h^{2+\alpha}|\ln h|^\beta\big). \tag{14}$$

Letting $e_i = u(x_i) - u_i$ and subtracting (11) from (12), we obtain the error $\{e_i\}$ satisfies the equation

$$\begin{cases} e_0 = 0, \\ e_i = h\sum_{j=0}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j|\big)^\beta \big[k\big(x_i, x_j, u(x_j)\big) - k(x_i, x_j, u_j)\big] \\ \qquad + h w_{ii}\big[k\big(x_i, x_i, u(x_i)\big) - k(x_i, x_i, u_i)\big] \\ \qquad + E_{i,t}\big(x_i, t, u(t)\big), \quad 1 \leqslant i \leqslant N. \end{cases} \tag{15}$$

From (5) we get

$$\begin{aligned} |e_i| \leqslant Lh\sum_{j=1}^{i-1} w_{ij}(x_i - x_j)^\alpha \big(\ln|x_i - x_j|\big)^\beta |e_j| + Lh\,w_{ii}|e_i| + \big|E_{i,t}\big(x_i, t, u(t)\big)\big|, \\ 1 \leqslant i \leqslant N. \end{aligned} \tag{16}$$

Let h be so small, that $Lh\,w_{ii} \leqslant \frac{1}{2}$, then we easily derive that

$$\begin{cases} e_0 = 0, \\ |e_i| \leqslant 2Lh\sum_{j=1}^{i-1}(x_i - x_j)^\alpha \big(\ln|x_i - x_j|\big)^\beta |e_j| + 2\big|E_{i,t}\big(x_i, t, u(t)\big)\big|, \\ \quad 1 \leqslant i \leqslant N. \end{cases} \tag{17}$$

Let

$$A = \max_{1\leqslant i\leqslant N} \max_{a\leqslant t\leqslant b} \left|2E_{i,t}\big(x_i, t, u(t)\big)\right|,$$
$$B_{ij} = 2Lh\,(x_i - x_j)^\alpha \big(\ln|x_i - x_j|\big)^\beta, \quad \text{for } i > j,\ -1 < \alpha < 0,\ \beta = 0 \text{ or } 1, \tag{18}$$

then (17) can be simplified as follows

$$\begin{cases} |e_0| = 0, \\ |e_i| \leqslant A + \sum_{j=1}^{i-1} B_{ij}|e_j|, \quad 1 \leqslant i \leqslant N. \end{cases} \tag{19}$$

Thus the convergence and error estimate of the approximation equation (11) are reduced to estimate $\{|e_i|\}$ satisfying (19). Unfortunately if A and B_{ij} in (19) are defined by (18), then the discrete Gronwall inequality (2) does not hold. In this paper, replacing (2) we shall prove a new generalization of discrete Gronwall inequality.

2. Theorem 1 and its proof

Theorem 1. *If A and B_{ij} are defined by* (18) *and $\{e_i\}$ satisfies the inequality* (19), *then there is a positive constant c independent of h, such that*

$$\max_{0\leqslant i\leqslant N} |e_i| \leqslant ch^{2+\alpha}(\ln h)^\beta. \tag{20}$$

Proof. Since $e_0 = 0$, successively substituting the right side of (19) when $i = n-1, \ldots, 1$ into

$$|e_n| \leqslant A + \sum_{j=1}^{n-1} B_{nj}|e_j|,$$

we derive

$$|e_n| \leqslant A + A\sum_{j_1=1}^{n-1} B_{nj_1} + A\sum_{j_2=1}^{n-1} B_{nj_2}\sum_{j_1=1}^{j_2-1} B_{j_2 j_1} + \cdots + A\sum_{j_{n-1}=1}^{n-1} B_{nj_{n-1}}\sum_{j_{n-2}=1}^{j_{n-1}-1} B_{j_{n-1}j_{n-2}}\cdots\sum_{j_1=1}^{j_2-1} B_{j_2 j_1}. \tag{21}$$

In order to estimate $\{|e_n|\}$, we use the following simple inequality: if a nonnegative function $f(x)$ is monotone on $[0, n]$, then

$$\sum_{i=1}^{n-1} f(i) \leqslant \int_0^n f(x)\,dx. \tag{22}$$

First we consider the case 1 that $\beta = 0$, i.e., $B_{ij} = 2Lh(x_i - x_j)^\alpha$. Since x^α is monotone and nonnegative, using (22) and setting $x = j_2 y$, we get

$$\sum_{j_1=1}^{j_2-1} B_{j_2 j_1} = 2Lh^{1+\alpha} \sum_{j_1=1}^{j_2-1} (j_2 - j_1)^{\alpha}$$
$$\leqslant 2Lh^{1+\alpha} \int_0^{j_2} (j_2 - x)^{\alpha}\, dx = 2Lh^{1+\alpha} B(1+\alpha, 1)\, j_2^{1+\alpha}, \tag{23}$$

where $B(r, s)$ denote Beta function. Similarly it holds that

$$F_{n,k} = A \sum_{j_k=1}^{n-1} B_{n j_k} \sum_{j_{k-1}=1}^{j_k-1} B_{j_k j_{k-1}} \cdots \sum_{j_1=1}^{j_2-1} B_{j_2 j_1}$$
$$\leqslant A\bigl(2Lh^{1+\alpha}\bigr)^k n^{k(1+\alpha)} B(1+\alpha, 1) \cdots B\bigl(1+\alpha, (k-1)(1+\alpha)+1\bigr). \tag{24}$$

However

$$B\bigl(1+\alpha, m(1+\alpha)+1\bigr) = \frac{\Gamma(1+\alpha)\Gamma(m(1+\alpha)+1)}{\Gamma((m+1)(1+\alpha)+1)}$$
$$= \frac{m}{m+1}\Gamma(1+\alpha)\frac{\Gamma(m(1+\alpha))}{\Gamma((m+1)(1+\alpha))}. \tag{25}$$

By Stirling formula there exists $\theta_z \in (0, 1)$, such that

$$\Gamma(z) = \sqrt{2}\pi z^{z-\frac{1}{2}} e^{-z} e^{\frac{\theta_z}{12z}}.$$

Letting

$$s = 1 + \alpha,$$

we get that

$$\frac{\Gamma(ms)}{\Gamma((m+1)s)} = \left(\frac{m}{m+1}\right)^{ms-\frac{1}{2}} \bigl[(m+1)s\bigr]^{-s} e^s \exp\left(\frac{\theta_m}{12ms} - \frac{\theta_{m+1}}{12(m+1)s}\right)$$
$$\leqslant e^{\frac{1}{12s}} \left(\frac{e}{s}\right)^s (m+1)^{-s}. \tag{26}$$

Substituting (26) into (25), we obtain

$$B(s, ms+1) \leqslant M(m+1)^{-s}, \tag{27}$$

where $M = \Gamma(s) e^{\frac{1}{12s}} (\frac{e}{s})^s$. Applying (27) to (24), we follow that

$$F_{n,k} \leqslant \frac{A(2L(b-a)^s M)^k}{(k!)^s} \leqslant A\frac{R^k}{(k!)^s}, \tag{28}$$

where $R = 2L(b-a)sM$. Thus after substituting (28) into (21), we obtain

$$|e_n| \leqslant A + A\sum_{k=1}^{n-1} \frac{R^k}{(k!)^s} < A\sum_{k=0}^{\infty} \frac{R^k}{(k!)^s} = HA, \tag{29}$$

where

$$H = \sum_{k=0}^{\infty} \frac{R^k}{(k!)^s} < \infty,$$

and H is a positive constant independent of h. However from (9) $A = \mathrm{O}(h^{2+\alpha})$, so there is a positive constant c independent of h satisfying

$$\max_{0 \leqslant i \leqslant N} |e_i| \leqslant ch^{2+\alpha}. \tag{30}$$

Secondly consider the case 2 that $\beta = 1$, i.e., $B_{ij} = 2Lh(x_i - x_j)^\alpha (\ln|x_i - x_j|)$. Note that since $\alpha > -1$, we can find such $\varepsilon > 0$, that $\alpha - \varepsilon > -1$. However by the inequality [8]

$$\bigl|\ln\bigl((i-j)h\bigr)\bigr| = -\ln\bigl((i-j)h\bigr) \leqslant \frac{((i-j)h)^{-\varepsilon}}{\varepsilon e},$$

we derive

$$|B_{ij}| \leqslant 2Lh\bigl[(i-j)h\bigr]^\alpha \frac{[(i-j)h]^{-\varepsilon}}{\varepsilon e} = \frac{2L}{\varepsilon e} h\bigl[(i-j)h\bigr]^{\alpha-\varepsilon}. \tag{31}$$

Setting

$$\alpha_1 = \alpha - \varepsilon, \qquad L_1 = \frac{L}{\varepsilon e},$$

and using the results of case 1, we can prove that

$$\max_{0 \leqslant i \leqslant N} |e_i| \leqslant ch^{2+\alpha} (\ln h)^\beta.$$

Thus the proof of Theorem 1 is completed. □

From Theorem 1 we easy prove the following corollary.

Corollary. *If $k(s,t,u)$ satisfies* (5)*, then the solutions $\{u_i\}$ of* (11) *converges to $\{u(x_i)\}$ as $h \to 0$, and there exists a constant c independent of h, such that when h is sufficiently small, the error has the estimate*

$$\max_{0 \leqslant i \leqslant N} |e_i| \leqslant \mathrm{ch}^{2+\alpha} (\ln h)^\beta. \tag{32}$$

3. A numerical example

Considering the second weakly singular Volterra equation

$$x(s) = \frac{11}{18} s^3 - \frac{1}{3} s^3 \ln s + s + \int_0^s \ln|s-t| x^2(t)\, dt.$$

We know the exact solution is $x(s) = s$, and taking the step width $h = 0.1, 0.05$, the results are listed in the following Table 1.

From the ratio of comparison in Table 1 we can know that the approximation has the accuracy order $\mathrm{O}(h^2 |\ln h|)$.

Table 1

t	0.7	0.8	0.9	1.0
$h = 0.1$_*error*	2.78E–3	2.93E–3	3.29E–3	3.22E–3
$h = 0.05$_*error*	7.99E–4	8.25E–4	8.45E–4	8.61E–4
ratio of comparison	0.284	0.281	0.257	0.267

Remark. After Gronwall's work [5], many authors generalized Gronwall's inequality and its discrete analogue (see [1,2,8,10,11]). The discrete analogues of the generalizations of Gronwall's inequality are often applied to the numerical treatment of differential equations and integral equations (see [9]), but the numerical treatment of weakly singular integral equations seems to be difficult.

Acknowledgment

The work was supported by the National Natural Science Foundation of China, No. 10171073.

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